Probability and Its Applications

D.J. Daley
D. Vere-Jones

An Introduction to the Theory of Point Processes

Volume II: General Theory and Structure

Second Edition

Springer

Probability and Its Applications

A Series of the Applied Probability Trust

Probability and Its Applications

Azencott et al.: Series of Irregular Observations. Forecasting and Model Building
Bass: Diffusions and Elliptic Operators
Bass: Probabilistic Techniques in Analysis
Berglund/Gentz: Noise-Induced Phenomena in Slow-Fast Dynamical Systems: A Sample-Paths Approach
Chen: Eigenvalues, Inequalities and Ergodic Theory
Costa/Fragoso/Marques: Discrete-Time Markov Jump Linear Systems
Daley/Vere-Jones: An Introduction to the Theory of Point Processes I: Elementary Theory and Methods, Second Edition
Daley/Vere-Jones: An Introduction to the Theory of Point Processes II: General Theory and Structure, Second Edition
de la Pena/Gine: Decoupling: From Dependence to Independence, Randomly Stopped Processes U-Statistics and Processes Martingales and Beyond
Del Moral: Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications
Durrett: Probability Models for DNA Sequence Evolution
Galambos/Simonelli: Bonferroni-Type Inequalities with Equations
Gani (ed.): The Craft of Probabilistic Modelling. A Collection of Personal Accounts
Guyon: Random Fields on a Network. Modeling, Statistics and Applications
Kallenberg: Foundations of Modern Probability, 2nd edition
Kallenberg: Probabilistic Symmetries and Invariance Principles
Last/Brandt: Marked Point Processes on the Real Line
Molchanov: Theory of Random Sets
Nualart: The Malliavin Calculus and Related Topics, 2nd edition
Shedler: Regeneration and Networks of Queues
Silvestrov: Limit Theorems for Randomly Stopped Stochastic Processes
Rachev/Rueschendorf: Mass Transportation Problems. Volume I: Theory
Rachev/Rueschendorf: Mass Transportation Problems. Volume II: Applications
Resnick: Extreme Values, Regular Variation and Point Processes
Thorisson: Coupling, Stationarity and Regeneration

D.J. Daley D. Vere-Jones

An Introduction to the Theory of Point Processes

Volume II: General Theory and Structure

Second Edition

Springer

D.J. Daley
Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra. ACT 0200, Australia
daryl@maths.anu.edu.au

D. Vere-Jones
School of Mathematics, Statistics
and Computing Science
Victoria University of Wellington
Wellington, New Zealand
David.Vere-Jones@mcs.vuw.ac.nz

Series Editors:

ISBN 978-0-387-21337-8

e-ISBN 978-0-387-49835-5

Library of Congress Control Number: 2007936157

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

springer.com

To Nola,
and in memory of Mary

Preface to Volume II, Second Edition

In this second volume, we set out a general framework for the theory of point processes, starting from their interpretation as random measures. The material represents a reorganized version of those parts of Chapters 6–14 of the first edition not already covered in Volume I, together with a significant amount of new material.

Contrary to our initial expectations, growth in the theoretical aspects of the subject has at least matched the growth in applications. Much of the original text has been substantially revised in order to present a more consistent treatment of marked as well as simple point processes. This applies particularly to the material on stationary processes in Chapter 12, the Palm theory covered in Chapter 13, and the discussion of martingales and conditional intensities in Chapter 14. Chapter 15, on spatial point processes, has also been significantly modified and extended. Essentially new sections include Sections 10.3 and 10.4 on point processes defined by Markov chains and Markov point processes in space; Sections 12.7 on long-range dependence and 12.8 on scale invariance and self-similarity; Sections 13.4 on marked point processes and convergence to equilibrium and 13.6 on fractal dimensions; Sections 14.6 on random time changes and 14.7 on Poisson embedding and convergence to equilibrium; much of the material in Sections 15.1–15.4 on spatial processes is substantially new or revised; and some recent material on point maps and point stationarity has been included in Section 13.3.

As in the first edition, much of the general theory has been developed in the context of a complete separable metric space (c.s.m.s. throughout this volume). Critical to this choice of context is the existence of a well-developed theory of measures on metric spaces, as set out, for example, in Parthasarathy

(1967) or Billingsley (1968). We use this theory at two levels. First, we establish results concerning the space[1] $\mathcal{M}^{\#}_{\mathcal{X}}$ and $\mathcal{N}^{\#}_{\mathcal{X}}$ of realizations of random measures and point processes, showing that these spaces themselves can be regarded as c.s.m.s.s, and paying particular attention to sample path properties such as the existence of atoms. Second, leaning on these results, we use the same framework to discuss the convergence of random measures and point processes. The fact that the same theory appears at both levels lends unity and economy to the development, although care needs to be taken in discriminating between the two levels.

The text of this volume necessarily assumes greater familiarity with aspects of measure theory and topology than was the case in Volume I, and the first two appendices at the end of Volume I are aimed at helping the reader in this regard. The third appendix reviews some of the material from martingale theory and the general theory of processes that underlies the discussion of predictability and conditional intensities in Chapter 14.

As was the case in Volume I, we are very much indebted to the friends, critics, reviewers, and readers who have supplied us with comments, suggestions, and corrections at various stages in the preparation of this volume. The list is too long to include in full, but we would like to mention in particular the continuing support and advice we have had from Robin Milne, Val Isham, Rick Schoenberg, Gunther Last, and our long-suffering colleagues in Canberra and Wellington. The patience and expertise of Springer Verlag, as mediated through our long-continued contacts with John Kimmel, are also very much appreciated.

Daryl Daley
Canberra, Australia

David Vere-Jones
Wellington, New Zealand

[1] In this edition we use $\mathcal{M}^{\#}_{\mathcal{X}}$ (and $\mathcal{N}^{\#}_{\mathcal{X}}$) to denote spaces of boundedly finite (counting) measures on $\mathcal{X}$ where in the first edition we used $\widehat{\mathcal{M}}_{\mathcal{X}}$ (and $\widehat{\mathcal{N}}_{\mathcal{X}}$), respectively.

Contents

Chapter Titles for Volume I

Principal Notation

Very little of the general notation used in Appendices 1–3 in Volume I is given below. Also, notation that is largely confined to one or two sections of the same chapter is mostly excluded, so that neither all the symbols used nor all the uses of the symbols shown are given. The repeated use of some symbols occurs as a result of point process theory embracing a variety of topics from the theory of stochastic processes. Generally, the particular interpretation of symbols with more than one use is clear from the context. Where they are given, page numbers indicate the first or significant use of the notation. Page numbers in slant font, such as *158*, refer to Volume I, but such references are not intended to be comprehensive.

Throughout the lists below, N denotes a point process, ξ a random measure (or sometimes, as on p. 358, a cumulative process on $\mathbb{R}_+$), and $\mathcal{X}$ a c.s.m.s.

Spaces

$\mathbb{C}$	complex numbers	
$\mathbb{R} = \mathbb{R}^1$	real line	
$\mathbb{R}_+, \mathbb{R}_0^+$	nonnegative numbers, positive numbers	358
$\mathbb{R}^d$	d-dimensional Euclidean space	
$\mathbb{S}$	circle group and its representation as $(0, 2\pi]$	
$\mathbb{U}_{2\alpha}^d$	d-dimensional cube of side length 2α and vertices $(\pm\,\alpha, \ldots, \pm\,\alpha)$	
$\mathbb{X}$	countable state space for Markov chain	96
$\mathbb{Z},\ \mathbb{Z}_+$	integers of $\mathbb{R}$, $\mathbb{R}_+$	
$\mathcal{X}$	state space of N or ξ; often $\mathcal{X} = \mathbb{R}^d$; always $\mathcal{X}$ is c.s.m.s. (complete separable metric space)	
Ω	space of probability elements ω	
$\mathcal{E}$	measurable sets in probability space	
$(\Omega, \mathcal{E}, \mathcal{P})$	basic probability space on which N and ξ are defined	*158*, 7
$\mathcal{X}^{(n)}$	n-fold product space $\mathcal{X} \times \cdots \times \mathcal{X}$	*129*
$\mathcal{X}^{\cup}$	$= \mathcal{X}^{(0)} \cup \mathcal{X}^{(1)} \cup \cdots$	*129*

$\mathcal{B}(\mathcal{X})$	Borel σ-field generated by open spheres of $\mathcal{X}$	*34*
$\mathcal{B}_{\mathcal{X}}$	$= \mathcal{B}(\mathcal{X}),\ \ \mathcal{B} = \mathcal{B}_{\mathbb{R}} = \mathcal{B}(\mathbb{R})$	*34, 374*
$\mathcal{B}_{\mathcal{X}}^{(n)} = \mathcal{B}(\mathcal{X}^{(n)})$	product σ-field on product space $\mathcal{X}^{(n)}$	*129*
$\mathrm{BM}(\mathcal{X})$	bounded measurable functions of bounded support	*161*, 52
$\mathrm{BM}_+(\mathcal{X})$	nonnegative functions $f \in \mathrm{BM}(\mathcal{X})$	57
$\overline{\mathrm{BM}_+}(\mathcal{X})$	limits of monotone sequences from $\mathrm{BM}_+(\mathcal{X})$	57
$\mathcal{K}$	mark space for marked point process (MPP)	*194*, 7
$\mathcal{M}_{\mathcal{X}}\ (\mathcal{N}_{\mathcal{X}})$	totally finite (counting) measures on $\mathcal{X}$	*158*, 3
$\mathcal{M}_{\mathcal{X}}^{\#}$	boundedly finite measures on $\mathcal{X}$	*158*, 3
$\mathcal{N}_{\mathcal{X}}^{\#}$	boundedly finite counting measures on $\mathcal{X}$	*131*, 3
$\mathcal{N}_0^{\#}$	$= \mathcal{N}_{\mathcal{X}}^{\#} \setminus \{\emptyset\}$	90
$\mathcal{N}_{\mathcal{X}}^{\#*}$	simple counting measures in $\mathcal{N}_{\mathcal{X}}^{\#}$	24
$\mathcal{N}_0, \mathcal{N}_{\mathcal{X}}^{\#*}$	subset of $\mathcal{N}_{\mathcal{X}}^{\#*}$ with $N\{0\} > 0$ FIX !!!	24, 290
$\mathcal{S}^+$	doubly infinite sequences of positive numbers $\{t_0, t_{\pm1}, t_{\pm2}, \dots\}$ with $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} t_{-n} = \infty$	14
$\mathcal{U}$	linear space; complex-valued Borel measurable functions ζ on $\mathcal{X}$ with $\lvert\zeta\rvert \le 1$	52; 57
$\mathcal{U} \otimes \mathcal{V}$	product topology on product space $\mathcal{X} \times \mathcal{Y}$ of topological spaces $(\mathcal{X},\mathcal{U})$, $(\mathcal{Y},\mathcal{V})$	*378*
$\mathcal{V} = \mathcal{V}(\mathcal{X})$	$[0,1]$-valued measurable functions $h(\cdot)$ with $1 - h(\cdot)$ of bounded support in $\mathcal{X}$	59
$\mathcal{V}_0(\mathcal{X})$	$= \{h \in \mathcal{V}(\mathcal{X})\colon \inf_x h(x) > 0\}$, i.e., $-\log h \in \mathrm{BM}_+(\mathcal{X})$	59
$\overline{\mathcal{V}}(\mathcal{X})$	limits of monotone sequences from $\mathcal{V}(\mathcal{X})$	59
$\mathcal{W} = \mathcal{X} \times \mathcal{M}_{\mathcal{X}}^{\#}$	product space supporting Campbell measure $C_{\mathcal{P}}$	269

General

Unless otherwise specified, $A \in \mathcal{B}_{\mathcal{X}}$, k and $n \in \mathbb{Z}_+$, t and $x \in \mathbb{R}$, $h \in \mathcal{V}(\mathcal{X})$, and $z \in \mathbb{C}$.

$\breve{\ }$	reduced measure (by factorization)	*160*, 183
$\#$	extension of concept from totally finite to boundedly finite measure space	*158*, viii
F^{n*}	n-fold convolution power of measure or d.f. F	*55*
$_{\mathrm{a}},\ _{\mathrm{g}}$	suffixes for atomic measure, ground process of MPP	4, 3
$\lVert\mu\rVert$	variation norm of (signed) measure μ	*374*
a.e. μ, μ-a.e.	almost everywhere with respect to measure μ	*376*
a.s., $\mathcal{P}$-a.s.	almost sure, $\mathcal{P}$-almost surely	*376*
$A(\cdot), A^{\mathcal{F}}(\cdot)$	$\mathcal{F}$-compensator for ξ on $\mathbb{R}_+$	358
$A^{(n)}$	n-fold product set $A \times \cdots \times A$	*130*
$\mathcal{A}$	family of sets generating $\mathcal{B}$; semiring of bounded Borel sets generating $\mathcal{B}_{\mathcal{X}}$	*31, 368*
$c_k,\ c_{[k]}$	kth cumulant, kth factorial cumulant, of distribution $\{p_n\}$	*116*

$c(x) = c(y, y + x)$	covariance density of stationary mean square continuous process on $\mathbb{R}^d$	*160*, 69
$C_k(\cdot), C_{[k]}(\cdot)$	cumulant, factorial cumulant measure	*147*, 69
$C_2(A \times B) = \operatorname{cov}(\xi(A), \xi(B))$	covariance measure of ξ	*191*, 69
$\breve{C}_2(\cdot)$	reduced covariance measure of stationary N or ξ	*292*, 238
$C_{\mathcal{P}}, C^!_{\mathcal{P}}$	Campbell measure, modified Campbell measure	269, 270
$\breve{C}_{\mathcal{P}}(\cdot)$	reduced Campbell measure (= Palm measure)	287, 331
$\delta(\cdot)$	Dirac delta function	
$\delta_x(A)$	Dirac measure, $= \int_A \delta(u - x)\,\mathrm{d}u = I_A(x)$	*382*, 3
$\mathcal{D}_\alpha$	Dirichlet process	12
$\Delta F(x) = F(x) - F(x-)$	jump at x in right-continuous function F	*107*
Δ^L, Δ^R	left- and right-hand discontinuity operators	376
$F(\cdot\,;\,\cdot)$	finite-dimensional (fidi) distributions	*158*, 26
$\boldsymbol{\mathcal{F}}$; $\boldsymbol{\mathcal{F}}^\dagger$	history on $\mathbb{R}_+$; $\mathbb{R}$	*236*, 356; 394
$\Phi(\cdot)$	characteristic functional	*15*, 54
$G[h]$ $(h \in \mathcal{V})$	probability generating functional (p.g.fl.) of N	*144*, 59
$G_c[\cdot]$	p.g.fl. of cluster centre process N_c	*178*
$G_m[\cdot \mid x]$	p.g.fl. of cluster member process $N_m(\cdot \mid x)$	*178*, 192
G	expected information gain of stationary N on $\mathbb{R}$	*280*, 442
$\Gamma(\cdot)$	Bartlett spectrum for stationary ξ on $\mathbb{R}^d$	*304*, 205
$H(t)$	integrated hazard function (IHF) [$Q(t)$ in Vol.I]	*109*, 361
$H(\mathcal{P}; \mu)$	generalized entropy	*277*, 441
$\boldsymbol{\mathcal{H}}$; $\boldsymbol{\mathcal{H}}^\dagger$	internal history of ξ on $\mathbb{R}_+$; $\mathbb{R}$	*236*, 358; 395
$I_A(x) = \delta_x(A)$	indicator function of element x in set A	
$\mathcal{I}$	σ-field of events invariant under shift operator S_u	194
$J_n(A_1 \times \cdots \times A_n)$	Janossy measure	*124*
$J_n(\cdot \mid A)$	local Janossy measure	*137*, 73
$j_n(x_1, \ldots, x_n)$	Janossy density	*125*, 119, 506
K	compact set; generic Borel set in mark space $\mathcal{K}$	*371*, 8
$\ell(\cdot)$	Lebesgue measure in $\mathcal{B}(\mathbb{R}^d)$	*31*
$\ell_{\mathcal{K}}(\cdot)$	reference measure on mark space	401
$L[f]$ $(f \in BM_+(\mathcal{X}))$	Laplace functional of ξ	*161*, 57
$L_\xi[1 - h]$	p.g.fl. of Cox process directed by ξ	*170*
$\lambda(\cdot)$, λ	intensity measure of N, intensity of stationary N	44, *46*
$\lambda^*(t, \omega)$	conditional intensity function	*231*, 390
$\lambda^\dagger(t, \kappa, \omega)$	complete intensity function for stationary MPP on $\mathbb{R}$	394
$m_k(\cdot)$ $(m_{[k]}(\cdot))$	kth (factorial) moment density	*136*
$\breve{m}_2$, $\breve{M}_2$	reduced second-order moment density, measure, of stationary N	*289*
m_{g}	mean density of ground process N_{g} of MPP N	*198, 323*
$M(A)$	expectation measure $\mathrm{E}[\xi(A)]$	65
$M_k(\cdot)$	kth-order moment measure $\mathrm{E}[\xi^{(k)}(\cdot)]$	66
$N(A)$	number of points in A	*42*
$N(a, b]$, $= N((a, b])$	number of points in half-open interval $(a, b]$	*19, 42*

$N(t)$	$= N(0,t] = N((0,t])$	*42*
N_c	cluster centre process	*176*
$N_m(\cdot \mid x)$	cluster member or component process	*176*
N_{g}	ground process of MPP	*194*, 7
N^*	support counting measure of N	4
$\{(p_n, \Pi_n)\}$	probability measure elements for finite point process	*123*
$P = (p_{ij})$	matrix of one-step transition probabilities p_{ij} of discrete-time Markov chain on countable state space $\mathbb{X}$	96
$P(z)$	probability generating function (p.g.f.) of distribution $\{p_n\}$	10, *115*
$P_0(A)$	avoidance function	*31*, 33
$\mathcal{P}$	probability measure of N or ξ on c.s.m.s. $\mathcal{X}$	*158*, 6
$\mathcal{P}_0(\cdot)$	Palm distribution for stationary N or ξ on $\mathbb{R}$	288
$\overline{\mathcal{P}}_0$	averaged (= mean) Palm measure for stationary MPP	319
$\mathcal{P}_{(0,\kappa)}(\cdot)$	Palm measure for $\kappa \in \mathcal{K}$	318
$\mathcal{P}_{x,\kappa}(\cdot)$	local Palm measure for $(x,\kappa) \in \mathcal{X} \times \mathcal{K}$	318
$\{\pi_k\}$	stationary distribution for (p_{ij})	96
$\emptyset$, $\emptyset(\cdot)$	empty set; null measure	17; 88, 292
$Q = (q_{ij})$	Q-matrix of transition rates q_{ij} for continuous-time Markov chain on countable state space $\mathbb{X}$	97
$\rho(x,y)$	metric for x, y in metric space	*370*
$\rho(y \mid \mathbf{x})$	Papangelou (conditional) intensity	120, 506
$\{S_i\}$	nested bounded sets, $S_i \uparrow \mathcal{X}$ $(i \to \infty)$	16
$\{S_n\}$	random walk, sequence of partial sums	*66*
$S_r(x)$	sphere of radius r, centre x, in metric space $\mathcal{X}$	*371*, 5
S_r	$= S_r(0)$	459
$\{t_i(N)\}, \{t_i\}$	successive points of N on $\mathbb{R}$, $t_{-1} < 0 \le t_0$	15
$\{\tau_i\}$	intervals between points of N on $\mathbb{R}$, $\tau_i = t_i - t_{i-1}$	15
$\mathcal{T} = \{\mathcal{T}_n\} = \{\{A_{ni}\}\}$	dissecting system of nested partitions	*382*, 10
	tiling	16
$\mathcal{T}_\infty$	tail σ-algebra of process on $\mathbb{R}^d$	208
$U(A) = \mathrm{E}[N(A)]$	renewal measure	*67*

Concordance of Statements from the First Edition

The table below lists the identifying number of formal statements of the first edition (1988) of this book and their identification in both volumes of this second edition.

1988 edition	this edition	1988 edition	this edition
2.2.I–III	2.2.I–III	7.1.I–II	9.1.II–III
2.3.III	2.3.I	7.1.III	9.1.VII
2.4.I–II	2.4.I–II	7.1.IV–VI	9.1.IV
2.4.V–VIII	2.4.III–VI	7.1.VII	9.1.V
3.2.I–6.V	3.2.I–6.V	7.1.VIII and 6	9.1.VIII
		7.1.IX–X	9.1.XV, XII
4.2.I–6.V	4.2.I–6.V	7.1.XI	9.2.X
5.2.I–VII	5.2.I–VII	7.1.XII–XIII	6.4.I(a)–(b)
5.3.I–III	5.3.I–III	7.2.I	9.3.VII
5.4.I–III	5.4.I–III	7.2.II	9.3.VIII–IX
5.4.IV–VI	5.4.V–VII	7.2.III	9.3.VIII
5.5.I	5.5.I	7.2.IV	9.3.X
6.1.I	9.1.I	7.2.V–VIII	9.3.XII–XV
6.1.II and 7	9.1.VI	7.3.I–V	9.2.XI–XV
6.1.III–IV and 7	9.1.VIII–IX	7.4.I–II	9.4.IV–V
6.1.V–VII and 7	9.1.XIV–XVI	7.4.III	9.5.VI
6.2.I–IX	9.2.I–IX	7.4.IV–V	9.4.VII–VIII
6.3.I–VI	9.3.I–VI	7.4.VI	9.5.II
6.3.VII–IX	10.1.II–IV	7.4.VII	9.4.IX
6.4.I–II	9.4.I–II	8.1.I	(6.1.13)
6.4.III	9.5.I	8.1.II	6.1.II, IV
6.4.IV–V	9.5.IV–V	8.2.I	6.3.I
6.4.VI	9.4.III	8.2.II	6.3.II, (6.3.6)
6.4.VII–IX	9.4.VI–VIII	8.2.III	Ex.12.1.6

1988 edition	this edition
8.2.IV	(6.3.6)
8.3.I–III	6.3.III–V
8.4.I–VIII	10.2.I–VIII
8.5.I–III	6.2.II
9.1.I–VII	11.1.II–VIII
9.1.IX	11.1.IX
9.1.X	11.1.XI
9.1.XI	12.3.X
Ex.9.1.3	11.1.X
9.2.I–VI	11.2.I–VI
(9.2.12)	11.2.VII
9.2.VII	10.2.IX
9.2.VIII	11.2.VIII
9.3.I–V	11.3.I–V
9.4.I–V	11.4.I–V
9.4.VI–IX	13.6.I–IV
10.1.I–III	12.1.I–III
10.1.IV	12.1.VI
10.1.V–VI	12.4.I–II
10.2.I–V	12.2.I–V
10.2.VI	12.2.IV
10.2.VII–VIII	12.2.VI–VII
10.3.I–IX	12.3.I–IX
10.3.X–XII	12.4.III–V
10.4.I–II	12.6.I–II
10.4.III	12.1.IV
10.4.IV–VII	12.6.III–VI
10.5.II–III	15.2.I–II
10.6.I–VIII	15.3.I–VIII
11.1.I–V	8.6.I–V
11.2.I–II	8.2.I–II
11.3.I–VIII	8.4.I–VIII
11.4.I–IV	8.5.I–IV
11.4.V–VI	8.5.VI–VII
12.1.I–III	13.1.I–III
12.1.IV–VI	13.1.V–VII
12.2.I	13.2.I
12.2.II–V	13.2.III–VI

1988 edition	this edition
12.2.VI–VIII	13.4.I–III
12.3.I	13.1.IV
12.3.II–VI	13.3.I–V
12.4.I	13.4.IV
12.4.II	Ex.13.4.7
12.4.III	13.4.V
12.4.IV–VI	13.4.VII–IX
13.1.I–III	7.1.I–III
13.1.IV–VI	7.2.I–III
13.1.VII	7.1.IV
13.2.I–II	14.1.I–II
Ex.13.2(b)	14.1.III
13.2.III–IV	14.1.IV–V
13.2.V	14.1.VII
13.2.VI	14.2.I
13.2.VII–IX	14.2.VII
13.3.I	14.2.II
13.3.II–IV	14.3.I–III
13.3.V–VIII	14.4.I–IV
13.3.IX–XI	14.5.I–III
13.4.I–III	14.6.I–III
13.4.III	7.4.I
13.4.IV	14.2.VIII
13.4.V	14.2.VII
13.4.VI	14.2.IX
13.5.I–II	pp. 394–396
13.5.III	14.3.IV
13.5.IV–V	14.3.V–VI
13.5.VI	14.8.I
13.5.VII–IX	14.8.V–VII
13.5.X	(14.8.9)
13.5.XI	14.8.VIII
14.2.I–V	15.6.I–III, V–VI
14.2.VI–VII	15.7.I–II
14.3.I–III	15.7.III–V

Appendix identical except for

A2.1.IV	A1.6.I
A2.1.V–VI	A2.1.IV–V

CHAPTER 9

Basic Theory of Random Measures and Point Processes

This chapter sets out a framework for developing point process theory as part of a general theory of random measures. This framework was developed during the 1940s and 1950s, and reached a definitive form in the now classic treatments by Moyal (1962) and Harris (1963). It still provides the basic framework for describing point processes both on the line and in higher-dimensional spaces, including especially the treatment of finite-dimensional distributions, moment structure, and generating functionals. In the intervening decades, many important alternative approaches have been developed for more specialized classes of processes, particularly those with an evolutionary structure, and we come to some at least of these in later chapters.

As far as is convenient, we develop the theory in a dual setting, stating results for general random measures alongside the more specific more clearly the features that are peculiar to point processes. Thus, for results that hold in this unified context, proofs are usually given only in the former, more general, setting.

Furthermore, the setting for point processes also handles many of the topics of this chapter for marked point processes (MPPs): an MPP in state space $\mathcal{X}$ with mark space $\mathcal{K}$ can be regarded as a point process on the product space $\mathcal{X} \times \mathcal{K}$ so far as fidi distributions, generating functionals, and moment measures are concerned. It is only when we consider particular cases, such as Poisson and compound Poisson processes or purely atomic random measures, that distinctions begin to emerge, and become more apparent as we move to discuss stationary processes in Chapter 12 and Palm theory and martingale properties in Chapters 13 and 14.

The other major approach to point process theory is through random sequences of points. We note that this is equivalent to our approach through random measures, at least in our setting that includes point processes in finite-dimensional Euclidean space $\mathbb{R}^d$.

Section 9.1 sets out some basic definitions and illustrates them with a variety of examples. The second section introduces the finite-dimensional (fidi) distributions and establishes both basic existence theorems and a version of Rényi's theorem that simple point processes are completely characterized by the behaviour of the *avoidance function* (vacuity function, empty space function), viz. the probability $P_0(A) \equiv \mathcal{P}\{N(A) = 0\}$ over a suitably rich class of Borel sets A. Section 9.3 is concerned with the sample path properties of random measures and point processes, and includes a detailed discussion of simplicity (orderliness) for point processes. The final two sections treat generating functionals and moment properties, extending the treatment for finite point processes given in Chapter 5.

9.1. Definitions and Examples

Let $\mathcal{X}$ be an arbitrary complete separable metric space (c.s.m.s.) and $\mathcal{B}_\mathcal{X} = \mathcal{B}(\mathcal{X})$ the σ-field of its Borel sets. Except for case (v) of Definition 9.1.II, all the measures that we consider on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ are required to satisfy the boundedness condition set out in Definition 9.1.I. It extends to general measures the property required of counting measures in Volume I, that bounded sets have finite counting measure and hence, as point sets, they contain only finitely many points and therefore have no finite accumulation points.

Definition 9.1.I. *A Borel measure μ on the c.s.m.s. $\mathcal{X}$ is* boundedly finite *if $\mu(A) < \infty$ for every bounded Borel set A.*

This constraint is incorporated into the definitions below of the spaces which form the main arena for the analysis in this volume. They incorporate the basic metric properties of spaces of measures summarized in Appendix A2 of Volume I. In particular we use from that appendix the following.

(1) The concept of *weak convergence* of totally finite measures on $\mathcal{X}$, namely that $\mu_n \to \mu$ weakly if and only if $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ for all bounded continuous f on $\mathcal{X}$ (see Section A2.3).

(2) The extension of weak convergence of totally finite measures to $w^\#$ (weak-hash) convergence of boundedly finite measures defined by $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ for all bounded continuous f on $\mathcal{X}$ vanishing outside a bounded set (Section A2.6).

(3) The fact that both weak and weak-hash convergence are equivalent to forms of metric convergence, namely convergence in the Prohorov metric

at equation (A2.5.1) and its extension to the boundedly finite case given by equation (A2.6.1), respectively.

Exercise 9.1.1 shows that for sequences of totally finite measures, weak and weak-hash convergence are not equivalent.

Many of our results are concerned with one or other of the first two spaces defined below. Both are closed in the sense of the $w^{\#}$-topology referred to above, and in fact are c.s.m.s.s in their own right (Proposition 9.1.IV). At the same time it is convenient to introduce four further families of measures which play an important role in the sequel.

Definition 9.1.II.

(i) $\mathcal{M}_{\mathcal{X}}^{\#}$ *is the space of all boundedly finite measures on* $\mathcal{B}_{\mathcal{X}}$.

(ii) $\mathcal{N}_{\mathcal{X}}^{\#}$ *is the space of all boundedly finite integer-valued measures* $N \in \mathcal{M}_{\mathcal{X}}^{\#}$, *called* counting measures *for short.*

(iii) $\mathcal{N}_{\mathcal{X}}^{\#*}$ *is the family of all* simple *counting measures, consisting of all those elements of* $\mathcal{N}_{\mathcal{X}}^{\#}$ *for which*

$$N\{x\} \equiv N(\{x\}) = 0 \text{ or } 1 \quad (\text{all } x \in \mathcal{X}). \tag{9.1.1}$$

(iv) $\mathcal{N}_{\mathcal{X}\times\mathcal{K}}^{\#\mathrm{g}}$ *is the family of all boundedly finite counting measures defined on the product space* $\mathcal{B}(\mathcal{X}\times\mathcal{K})$, *where* $\mathcal{K}$ *is a c.s.m.s. of marks, subject to the additional requirement that the* ground measure N_{g} *defined by*

$$N_{\mathrm{g}}(A) \equiv N(A\times\mathcal{K}) \quad (\text{all } A \in \mathcal{B}_{\mathcal{X}}) \tag{9.1.2}$$

is a boundedly finite simple counting measure, i.e. $N_{\mathrm{g}} \in \mathcal{N}_{\mathcal{X}}^{\#*}$.

(v) $\mathcal{M}_{\mathcal{X},\mathrm{a}}^{\#}$ *is the family of boundedly finite purely atomic measures* $\xi \in \mathcal{M}_{\mathcal{X}}^{\#}$.

(vi) $\mathcal{M}_{\mathcal{X}}$ *(respectively,* $\mathcal{N}_{\mathcal{X}}$*) is the family of all totally finite (integer-valued) measures on* $\mathcal{B}_{\mathcal{X}}$.

We introduce the family $\mathcal{N}_{\mathcal{X}\times\mathcal{K}}^{\#\mathrm{g}}$ to accommodate our Definition 9.1.VI(iv) of a marked point process (MPP) (as a process on $\mathcal{X}$ with marks in $\mathcal{K}$). In it we require the ground process N_{g} to be both simple and boundedly finite. Note that in general a simple boundedly finite counting measure on $\mathcal{B}_{\mathcal{X}\times\mathcal{K}}$ need not be an element of this family $\mathcal{N}_{\mathcal{X}\times\mathcal{K}}^{\#\mathrm{g}}$. For example, taking $\mathcal{X} = \mathcal{K} = \mathbb{R}$, realizations of a homogeneous Poisson process on the plane would have ground process elements failing to be members of $\mathcal{N}_{\mathbb{R}}^{\#}$. See also Exercises 9.1.3 and 9.1.6.

Note also that although a purely atomic boundedly finite measure can have at most countably many atoms, these atoms may have accumulation points, so representing such measures as a countable set $\{(x_i,\kappa_i)\}$ of pairs of locations and sizes of the atoms can give a counting measure on $\mathcal{X}\times\mathbb{R}_+$ that need not be in either $\mathcal{N}_{\mathcal{X}\times\mathbb{R}_+}^{\#\mathrm{g}}$ nor even $\mathcal{N}_{\mathcal{X}\times\mathbb{R}_+}^{\#}$ [cf. Proposition 9.1.III(v) below].

In investigating the closure properties of $\mathcal{M}_{\mathcal{X}}^{\#}$ and $\mathcal{N}_{\mathcal{X}}^{\#}$ (Lemma 9.1.V below), we use *Dirac measures* (see Section A1.6) defined for every $x \in \mathcal{X}$ by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in \text{Borel set } A, \\ 0 & \text{otherwise.} \end{cases} \tag{9.1.3}$$